the maximum droplet diameters are the same. Clearly, fairly different h(s) correspond to virtually identical distributions. There is a more marked effect from the choice of maximum size. In all our experiments, the maximum diameter was taken as the value of s at which the number of pulses constituted 0.02% of the maximum.

The reliability of the data was examined by recording the distribution by deposition; the drops were trapped in a thin layer of silicone oil on a plate of dimensions $2 \times 3 \cdot 10^{-6}$ m, the total number of drops being 1240. Curve 1 in Fig. 4 is from the Wicks–Dukler method operating with the above circuit, while curve 2 is from the deposition method. The agreement is satisfactory, particularly in the region of the mode.

NOTATION

$Re = c\delta\rho/\mu$, Reynolds number; c, drop velocity; δ, film thickness; ρ, density of liquid; μ, dynamic viscosity coefficient of a liquid; f(D), size distribution; $f_V(D)$, volume distribution; F, cross-sectional area of flow; D, drop diameter; D_m, maximum drop diameter; s, distance between electrodes; h, pulse frequency; A, α, approximation coefficients.

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SOLID PARTICLE IMPURITY PROPAGATION IN A FLUID FLOW IN A PIPE

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A longitudinal diffusion model is proposed for planar flow in which the fluid particles and the impurities have different velocities.

The process of solid particle impurity propagation in an incompressible fluid flow in a flat pipe is investigated. The turbulent and convective diffusion processes as well as the settling of the particles under the effect of gravity result in the impurity concentration varying in both the stream depth and along it. Since the question of impurity propagation along the stream is of special interest for applications, a derivation is given in this paper for a one-dimensional diffusion mixing model to determine the mean particle concentration over the stream section. The turbulent and convective diffusion mechanisms, particularly the velocity distribution in the stream, are taken into account in such a model by an effective coefficient for which an expression is found in terms of the local velocity field characteristics. The proposed one-dimensional diffusion model refers to streams in which the fluid and impurity particles have different average velocities. The velocity of convective transport in this model does not equal the mean stream velocity, which distin-

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guishes this model from those in [1, 2]. We follow the methodology in the known work of Taylor [3].

Before presenting the derivation of the equation describing the impurity particle propagation in the stream, we make some fundamental assumptions within whose framework the present investigation is executed.

We consider the particles in the stream to be considered as a continuous distributed continuum possessing density and velocity. We assume the size of the suspended particles to be small compared with the characteristic turbulence scales. We limit ourselves to the consideration of that case when the volume concentration s is low despite the fact that a large quantity of particles is contained in the stream.

Let us consider the suspension-carrying stream as a two-phase system. Let us write the continuity equation for each of the phases

$$\frac{\partial\rho_1}{\partial t}+\frac{\partial\rho_1 v_{1\alpha}}{\partial x_\alpha}=0, \tag{1}$$

$$\frac{\partial\rho_2}{\partial t}+\frac{\partial\rho_2 v_{2\alpha}}{\partial x_\alpha}=0,\quad \alpha=1,2,3.$$

By introducing the true fluid and particle material densities d_1 and d_2, we can express the densities of the separate phases in terms of the parameter s

$$\rho_1=(1-s)\,d_1,\quad \rho_2=sd_2.$$

Then the second equation in system (1) becomes

$$\frac{\partial s}{\partial t}+\frac{\partial s v_{2\alpha}}{\partial x_\alpha}=0. \tag{2}$$

We represent all the quantities in the form of sums of their average values and the pulsating components in conformity with the method of describing turbulent flows

$$s=\bar{s}+s',\quad v_{2\alpha}=\bar{v}_{2\alpha}+v'_{2\alpha}. \tag{3}$$

Then (2) can be rewritten as follows

$$\frac{\partial s}{\partial t}+\frac{\partial s\,(\bar{v}_{2\alpha}-\bar{v}_{1\alpha})}{\partial x_\alpha}+\frac{\partial\,\bar{s}\bar{v}_{1\alpha}}{\partial x_\alpha}=-\frac{\partial\,\overline{s'v'_{2\alpha}}}{\partial x_\alpha}. \tag{4}$$

Here the difference $(\bar{v}_{2\alpha}-\bar{v}_{1\alpha})$ is the relative velocity of the impurities in the carrying medium.

Because of the assumptions made above about the particle size, it can be assumed that the horizontal average velocities of the solid particles and the fluid agree, while the vertical velocities differ by a certain quantity a, i.e.,

$$\bar{v}_{2\alpha}=\bar{v}_{1\alpha}-a\delta_{\alpha 3},\quad (\delta_{\alpha\beta}=0,\ \alpha\neq\beta,\ \delta_{\alpha\alpha}=1). \tag{5}$$

Here the quantity a is called the hydraulic lumpiness. If the impurity particles are more or less identical in size, then the quantity a can be considered constant.

Taking account of (5) we obtain

$$\frac{\partial\bar{s}}{\partial t}-a\frac{\partial\bar{s}}{\partial x_3}+\frac{\partial\,\bar{s}\bar{v}_{1\alpha}}{\partial x_\alpha}=-\frac{\partial\,\overline{s'v'_{2\alpha}}}{\partial x_\alpha}. \tag{6}$$

A further conversion of (6) can be accomplished if the correlation between the parameters s' and $v'_{2\alpha}$ is taken proportional to the gradient of the function $\bar{s}$ in conformity with the Boussinesq hypothesis:

$$\overline{s'v'_{2\alpha}}=\varepsilon_{\alpha\beta}\frac{\partial\bar{s}}{\partial x_\beta}. \tag{7}$$

In this formula $\varepsilon_{\alpha\beta}$ is the diffusion coefficients tensor which has the following form in the case of isotropic diffusion

$$\varepsilon_{\alpha\beta} = -\varepsilon\delta_{\alpha\beta}.$$

After having used (7), Eq. (6) is written as follows

$$\frac{\partial \bar{s}}{\partial t} - a\frac{\partial \bar{s}}{\partial x_3} + \frac{\partial \overline{s v_{1\alpha}}}{\partial x_\alpha} = -\frac{\partial}{\partial x_\alpha}\left(\varepsilon_{\alpha\beta}\frac{\partial \bar{s}}{\partial x_\beta}\right). \tag{8}$$

Let us examine impurity transport in a plane-parallel stream of depth H. We direct the z axis vertically upward and the x axis in the direction of stream motion. The stream velocity vector will have just one nonzero component $v_x(z)$, that is, $\mathbf{v}(v_x(z), 0, 0)$, in correspondence to the case under consideration. Taking this circumstance into account, (8) can be given the following form:

$$\frac{\partial \bar{s}}{\partial t} - a\frac{\partial \bar{s}}{\partial z} + v_x(z)\frac{\partial \bar{s}}{\partial x} = \frac{\partial}{\partial x}\left(\varepsilon\frac{\partial \bar{s}}{\partial x}\right) + \frac{\partial}{\partial z}\left(\varepsilon\frac{\partial \bar{s}}{\partial z}\right). \tag{9}$$

We take as initial condition that the function $\bar{s} = s_0(x)$ is known at t = 0 while there is no impurity flux at the stream boundaries z = 0 and z = H:

$$\varepsilon\frac{\partial \bar{s}}{\partial z} + a\bar{s} = 0.$$

We shall henceforth consider the longitudinal stream velocity and the transfer coefficient in Eq. (9) obtained not to differ from the corresponding parameters in a homogeneous stream without impurity particles. To a known degree this assumption has been justified earlier by the condition taken on the smallness of the particle volume concentration. The presence of the solid particles actually alters the structure of the turbulent stream and affects its characteristics. A derivation of the transport equation with such influence taken into account is given in [4].

Equation (9) reflects the fact that the impurity particle propagation in the stream is due to two causes: convective transport and turbulent diffusion. Both these processes can be taken into account within the framework of one-dimensional diffusion model by introducing an effective transfer coefficient whose magnitude depends on the stream velocity profile and the concentration distribution in an integral manner.

Let us use the procedure described in [5] to construct such a model and to derive the formula for the effective transfer coefficient.

We define the mean volume impurity concentration over a stream section by the formula

$$\Theta = \frac{1}{H}\int_0^H \bar{s}\,dz. \tag{10}$$

Let us multiply (9) by the quantity 1/H and let us integrate with respect to z between the limits 0 and H. We hence neglect the longitudinal turbulent transport which, as estimates show, yields a small contribution to the magnitude of the effective coefficient:

$$\frac{\partial \Theta}{\partial t} + U\frac{\partial \Theta}{\partial x} + \frac{1}{H}\int_0^H (v_x - U)(\bar{s} - \Theta)\,dz = 0. \tag{11}$$

In this formula U is the mean stream velocity

$$U = \frac{1}{H}\int_0^H v_x(z)\,dz. \tag{12}$$

Denoting the difference between the quantities $\bar{s}$ and Θ by Ψ, using (9) and (11) as well as the conditions on the stream boundaries, we have the following problem for the function Ψ

$$\frac{\partial \Psi}{\partial t} - a\frac{\partial \Psi}{\partial z} - \frac{\partial}{\partial z}\left(\varepsilon\frac{\partial \Psi}{\partial z}\right) = -U\frac{\partial \Psi}{\partial x} - (v_x - U)\frac{\partial \bar{s}}{\partial x} + \frac{1}{H}\frac{\partial}{\partial x}\int_0^H (v_x - U)\Psi\,dz, \tag{13}$$

for z = 0 and z = H

$$\varepsilon\frac{\partial\Psi}{\partial z}+a\Psi=-a\Theta. \tag{14}$$

Using successive approximations, we set the average value Θ in place of $\bar{s}$ in the right-hand side of (13). The foundation for such an approximation is the fact that for times much greater than the diffusion constant H^2/ε_0 (ε_0 is a typical value of the transport coefficient), when the length of the mixture domain becomes much greater than the characteristic linear dimension H, the impurity concentration over the stream depth is almost equilibrated, and only small deflections of the local concentrations from its average value exist. These deviations are due to the inhomogeneous convective impurity transport because of the velocity profile. We obtain as a result of the formulation

$$\frac{\partial\Psi}{\partial t}-a\frac{\partial\Psi}{\partial z}-\frac{\partial}{\partial z}\left(\varepsilon\frac{\partial\Psi}{\partial z}\right)=-(v_x-U)\frac{\partial\Theta}{\partial x}. \tag{15}$$

If the notation

$$r(z)=\exp\int_0^z\frac{a}{\varepsilon(z')}dz',\quad p(z)=\varepsilon(z)\,r(z)$$

is introduced, then the operator of differentiation with respect to z in the left-hand side of (15) can be represented in the self-adjoint form

$$L(\Psi)=\frac{1}{r(z)}\frac{\partial}{\partial z}\left[p(z)\frac{\partial\Psi}{\partial z}\right]. \tag{16}$$

Using this operator we represent (15) in the form

$$L(\Psi)=\frac{\partial\Psi}{\partial t}+f, \tag{17}$$

where

$$f=(v_x-U)\frac{\partial\Theta}{\partial x}.$$

Let us seek the solution of the equation obtained in the form of a series in the eigenfunctions $X_n(z)$:

$$\Psi(t,x,z)=\sum_{n=1}^{\infty}\frac{u_n(t,x)}{\|X_n\|^2}X_n(z),$$
$$u_n(t,x)=\frac{1}{H}\int_0^H r(z)\,\Psi X_n(z)\,dz, \tag{18}$$
$$\|X_n\|^2=\frac{1}{H}\int_0^H r(z)\,X_n^2(z)\,dz.$$

The eigenfunctions $X_n(z)$ satisfy the following Sturm–Liouville problem:

$$(pX_n')'+\lambda_n^2 rX_n=0,\quad X_n=X_n(z),$$
$$\varepsilon X_n'+aX_n=0,\quad z=0,\ z=H. \tag{19}$$

Here λ_n are eigenvalues of the problem mentioned.

Using (18), we write (11) in the form

$$\frac{\partial\Theta}{\partial t}+U\frac{\partial\Theta}{\partial x}+\frac{1}{H}\sum_{n=1}^{\infty}\frac{a_n}{\|X_n\|^2}\frac{\partial u_n(t,x)}{\partial x}=0, \tag{20}$$

where

$$\Theta=\Theta(t,x),\quad a_n=\int_0^H(v_x-U)\,X_n(z)\,dz.$$

To determine the unknown functions $u_n(t, x)$ which are the coefficients of the series (18), we multiply (17) by rX_n and integrate with respect to z between 0 and H. Taking account of the boundary conditions (14) and (19), we obtain an equation for the functions $u_n(t, x)$

$$\frac{\partial u_n}{\partial t} + \lambda_n^2 u_n = \frac{1}{H}(b_n \Theta - f_n), \tag{21}$$

$$u_n = u_n(t, x).$$

Here

$$b_n = a[r(0) X_n(0) - r(H) X_n(H)],$$

$$f_n = \int_0^H r(z) f X_n(z)\, dz = c_n \frac{\partial \Theta}{\partial x},$$

$$c_n = \int_0^H r(z)(v_x - U) X_n(z)\, dz.$$

Taking account of the zero initial condition for the function $u_n(t, x)$, the solution of this equation has the form

$$u_n(t, x) = -\frac{c_n}{H} \int_0^t \exp[-\lambda_n^2 (t-\tau)] \frac{\partial \Theta}{\partial x} d\tau + \frac{b_n}{H} \int_0^t \Theta \exp[-\lambda_n^2 (t-\tau)]\, d\tau. \tag{22}$$

Substituting the expression found for the function $u_n(t, x)$ into (20), we arrive at an integrodifferential equation to determine the mean value $\Theta(t, x)$ of the volume impurity concentration:

$$\frac{\partial \Theta}{\partial t} + U \frac{\partial \Theta}{\partial x} + \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{a_n b_n}{\|X_n\|^2} \int_0^t \exp[-\lambda_n^2 (t-\tau)] \frac{\partial \Theta}{\partial x} d\tau = \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{a_n c_n}{\|X_n\|^2} \int_0^t \exp[-\lambda_n^2 (t-\tau)] \frac{\partial^2 \Theta}{\partial x^2} d\tau, \tag{23}$$

$$\Theta = \Theta(t, x).$$

The equation obtained describes the distribution of the mean concentration $\Theta(t, x)$ over the stream depth. By solving it we can find not only the particle concentration distribution over the stream depth by using (18), but also

$$\bar{s} = \Theta + \sum_{n=1}^{\infty} \frac{u_n(t, x)}{\|X_n\|^2} X_n(z).$$

The asymptotic solution of (23) as $t \to \infty$ is of interest. The fact is that the asymptotic distribution of the impurity particle concentration in a stream is established, in practice, sufficiently rapidly after the beginning of the process, hence the solution of the corresponding equation can be used to describe the process in almost the whole time span. For $t \to \infty$ Eq. (23) simplifies considerably:

$$\frac{\partial \Theta}{\partial t} + V \frac{\partial \Theta}{\partial x} = K \frac{\partial^2 \Theta}{\partial x^2}. \tag{24}$$

Here

$$K = \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{a_n c_n}{\lambda_n^2 \|X_n\|^2} \tag{25}$$

and

$$V = U + \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{a_n b_n}{\lambda_n^2 \|X_n\|^2}. \tag{26}$$

Expression (24) is a one-dimensional diffusion equation in which K is the coefficient of longitudinal diffusion taking account of the convective and turbulent transport mechanisms, and V is the velocity of convective impurity particle transport. Both these parameters effectively take account of the local transport mechanisms and appear in the one-dimensional model as analogs of the corresponding parameters of the one-dimensional molecular diffusion equation. However, the magnitude of the effective coefficient exceeds the value of the molecular and turbulent diffusion coefficients many times. This is explained by the fact that the difference between the averaged and mean stream velocities is much greater than the velocity fluctuations and the inhomogeneity of the velocity field is the main reason for the increase in the length of the mixing domain.

If the impurity particles do not settle, i.e., the quantity a is zero, then the magnitude of the parameter b_n vanishes and, therefore, V = U, i.e, the velocity of convective transport of a substance agrees with the mean stream velocity. If the rate of particle settling differs from zero, then the convective impurity transport occurs at a velocity not equal to the mean stream velocity, where its difference becomes more noticeable with the growth in the hydraulic lumpiness of the particles. It can be shown that the velocity V is less than the mean stream velocity.

As a simple illustration, let us consider the problem of propagation of an impurity which has been introduced in a stream at the origin x = 0 at the time t = 0. This corresponds to the following initial and boundary conditions

$$\Theta(0,\ x) = s_0\delta(x),\quad \Theta(t, \pm\infty) = 0.$$

Here s_0 is the quantity of the impurity introduced, and $\delta(x)$ is the delta function.

The solution of (24) in a coordinate system moving with the velocity V has the form

$$\Theta(x,\ t) = \frac{s_0}{2\sqrt{\pi Kt}} \exp\left[-\frac{(x-Vt)^2}{4Kt}\right]. \tag{27}$$

It follows from this solution that the maximum value of the concentration of the settling impurities is shifted at a velocity different from the mean stream velocity. This circumstance should be taken into account in measuring the mean stream velocity by inserting impurities.

NOTATION

ρ_1, ρ_2, densities of liquid component and impurity particles; $v_{1\alpha}$ and $v_{2\alpha}$, liquid and impurity velocity components; d_1 and d_2, true component densities; s, volume concentration; $\bar{s}$ and s', averaged and fluctuating concentration components; $\bar{v}_\alpha$ and v'_α, averaged and pulsating velocity components; $\varepsilon_{\alpha\beta}$, tensor of the diffusion coefficients; a, hydraulic lumpiness; H, channel width; Θ, mean volume impurity concentration over the stream section; U, mean stream velocity; X_n, λ_n, eigenfunctions and eigenvalues of the Sturm–Liouville problem; K, effective coefficient of diffusion; t, time; x_α, space variable.

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